THE SCOTS COLLEGE

2012

HSC Task 3 – In Class Test
13th June 2012

Mathematics Extension 2

General Instructions
• Working time – 45 minutes
• Write using black or blue pen
Black pen is preferred
• Board-approved calculators may be used
• A table of standard integrals is provided at the back of this paper
• Show all necessary working

Total marks – 30

Learning Intentions:

Integration and Volumes

QUESTION ONE (15 MARKS) Use a SEPARATE writing booklet.

a) $\int sin^3 \theta \, d\theta$ *(2)*

b) $\int xe^x \, dx$ *(2)*

c) By using the substitution $t = tan\frac{\theta}{2}$, evaluate in exact form *(3)*

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{5 - 4cos\theta}$$

d) *(3)*

$$\int_{-2}^{2} \left(x\sqrt{4 - x^2} - \sqrt{4 - x^2}\right) dx$$

e) Given that

$$I_n = \int_1^e (1 - lnx)^n \, dx, (n = 1,2,3, \dots)$$

i. Show that $I_n = -1 + nI_{n-1}$ *(3)*

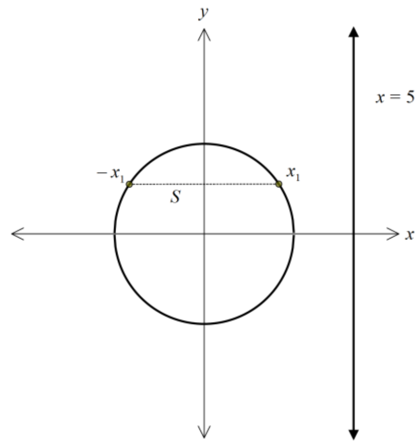
ii. Hence evaluate *(2)*

$$\int_1^e (1 - lnx)^3 \, dx$$

End of Question 1

QUESTION TWO (15 MARKS) Use a SEPARATE writing booklet.

a) The circle $x^2 + y^2 = 9$ is rotated about the line $x = 5$ to form a ring. When the circle is rotated, the line segment S at height y sweeps out an annulus. The coordinates of the end-points of S are x_1 and $-x_1$, where $x_1 = \sqrt{9 - y^2}$.



i. Show that the area of the annulus is equal to *(2)*

$$20\pi\sqrt{9 - y^2}$$

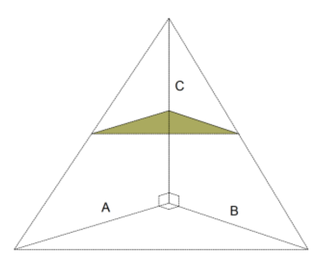
ii. Hence find the volume of the ring. *(2)*

b) *i.* Sketch region bounded by the curve $y = 4x - x^2$, $x = 1$, $y = x$ and the *y-axis*. *(1)*

ii. This region is rotated about the *y-axis.* Use the method of cylindrical shells to calculate the volume of the solid created. *(3)*

c) A particular solid has as its base the region bounded by the hyperbola $x^2 - y^2 = 1$ and the line $x = 2$. Cross-sections perpendicular to this base and the *x-axis* are semi-circles whose diameter are in the base. Find the volume of the solid. *(3)*

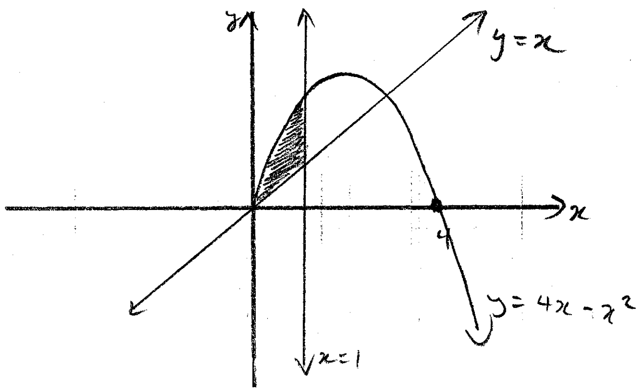
d) A tetrahedron has three mutually perpendicular faces and three mutually perpendicular edges of length A,B and C. *(4)*
By slicing parallel to the base and summing such slices, confirm the formula for the volume of a tetrahedron is $V = \frac{ABC}{6}$.

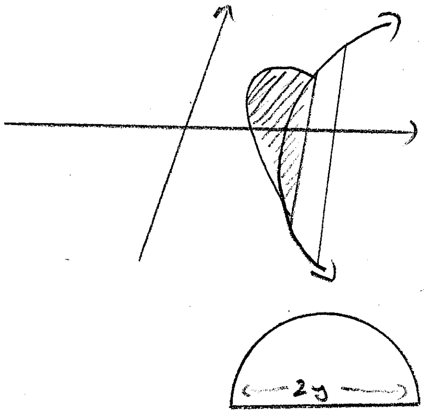
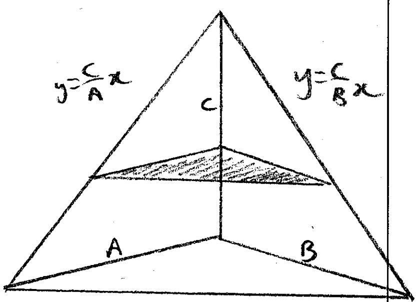


End of Paper

Q1	a)	$\int \sin^3\theta \, d\theta = \int \sin\theta(1-\cos^2\theta)\, d\theta$ $= \int (\sin\theta - \sin\theta\cos^2\theta)\, d\theta$ $= -\cos\theta + \frac{1}{3}\cos^3\theta + C$		1 Method 1 Solution
	b)	$\int x e^x \, dx = e^x (x-1) + \text{constant}$		
		Possible intermediate steps: $\int e^x x \, dx$ For the integrand $e^x x$, integrate by parts, $\int f\,dg = fg - \int g\,df$, where $f = x, \quad dg = e^x dx,$ $df = dx, \quad g = e^x$: 1 for by parts $= e^x x - \int e^x \, dx$ The integral of e^x is e^x: $= e^x x - e^x + \text{constant}$ Which is equal to: $= e^x (x-1) + \text{constant}$ 1 for solution		

	c)	$$\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos(x)}\,dx = \frac{2\pi}{9} \approx 0.698132$$		
		Possible intermediate steps: $\int \frac{1}{5-4\cos(x)}\,dx$ For the integrand $\frac{1}{5-4\cos(x)}$, substitute $u=\tan\left(\frac{x}{2}\right)$ and $du=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$. Then transform the integrand using the substitutions $\sin(x)=\frac{2u}{u^2+1}$, $\cos(x)=\frac{1-u^2}{u^2+1}$ and $dx=\frac{2\,du}{u^2+1}$: $=\int \frac{2}{(u^2+1)\left(5-\frac{4(1-u^2)}{u^2+1}\right)}\,du$ Simplify the integrand $\frac{2}{(u^2+1)\left(5-\frac{4(1-u^2)}{u^2+1}\right)}$ to get $\frac{2}{9u^2+1}$: $=\int \frac{2}{9u^2+1}\,du$ Factor out constants: $=2\int \frac{1}{9u^2+1}\,du$ The integral of $\frac{1}{9u^2+1}$ is $\frac{1}{3}\tan^{-1}(3u)$: $=\frac{2}{3}\tan^{-1}(3u)+\text{constant}$ Substitute back for $u=\tan\left(\frac{x}{2}\right)$: $=\frac{2}{3}\tan^{-1}\left(3\tan\left(\frac{x}{2}\right)\right)+\text{constant}$	1 mark set up 1 mark correct substitution	
		$$\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos(x)}\,dx = \frac{2\pi}{9} \approx 0.698132$$	1 mark solution	
	d)	$\int_{-2}^{2}\left(x\sqrt{4-x^2}-\sqrt{4-x^2}\right)dx$ $\underbrace{\int_{-2}^{2} x\sqrt{4-x^2}\,dx}_{\text{odd fn}} - \underbrace{\int_{-2}^{2}\sqrt{4-x^2}\,dx}_{\text{semi-circle}}$ radius 2 $0 \quad - \quad \frac{1}{2}\pi r^2$ $- \frac{1}{2}\pi\times 4 \quad = \quad -2\pi$	1 method 1 recognise odd & semicircle 1 solution	

	e) i)	$I_n = \int_1^e (1 - \ln x)^n \, dx, (n = 1,2,3, \dots)$ $= \left[x(1-\ln x)^n\right]_1^e - \int nx(1-\ln x)^{n-1} \cdot \left(-\frac{1}{x}\right) dx$ $= \left[0 - (1-0)^n\right] + n\int (1-\ln x)^{n-1} \, dx$ $I_n = -1 + nI_{n-1}$	1 method 1 progress 1 solution
	ii)	$\int_1^e (1-\ln x)^3 \, dx = -1 + 3I_2$ $= -1 + 3\left[-1 + 2I_1\right]$ $= -1 + 3\left[-1 + 2\int_1^e (1-\ln x) \, dx\right]$ $= -1 - 3 + 6\left[-1 + I_0\right]$ $= -10 + 6\int_1^e 1 \, dx$ $= -10 + 6\left[x\right]_1^e$ $= -10 + 6e - 6 = 6e - 16$	1 method 1 solution
Q2	a) i)	$A = \pi R^2 - \pi r^2$ where $R = 5 + x_1$, $r = 5 - x_1$ $A = \pi(5+x_1)^2 - \pi(5-x_1)^2$ $= \pi(5+x_1-5+x_1)(5+x_1+5-x_1)$ $= 20\pi x_1 = 20\pi\sqrt{9-y^2}$	1 set up 1 solution
	ii)	$V = 2\int_0^3 20\pi\sqrt{9-y^2} \, dy$ $= 40\pi \underbrace{\int_0^3 \sqrt{9-y^2}}_{\text{quarter circle}} \, dy$ $V = 40\pi\left(\frac{1}{4}\pi r^2\right)$ $r = 3$ $= 90\pi^2 \; u^3$	1 correct integral 1 solution
	b) i)	 	1 correct region

	Part	Solution	Marks	
	b) ii)	$\lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi r h \, \delta x$ $\lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi(x)(3x - x^2) \, \delta x$ $2\pi \int_0^1 3x^2 - x^3 \, dx$ $2\pi \left[\frac{3x^3}{3} - \frac{x^4}{4} \right]_0^1$ $2\pi \left[1 - \frac{1}{4} \right] = \frac{3}{2}\pi \, u^3$	1 correct use of cylindrical shells method 1 progress 1 solution	
	c)	$V = \frac{1}{2} \int_1^2 \pi r^2 \, dx$ $= \frac{1}{2} \int_1^2 \pi y^2 \, dx$ $= \frac{\pi}{2} \int_1^2 x^2 - 1 \, dx$ $= \frac{\pi}{2} \left[\frac{x^3}{3} - x \right]_1^2$ $= \frac{\pi}{2} \left[\frac{7}{3} - 1 \right]$ $= \frac{2\pi}{3} \, u^3$  $A = \frac{1}{2}\pi(y)^2$	1 correct area 1 set up & progress 1 solution	
	d)	$V = \int_0^c \frac{1}{2} \times b \times h \, dy$ $= \int_0^c \frac{1}{2} \left(\frac{A}{c} y \right) \left(\frac{B}{c} y \right) dy$ $= \frac{1}{2} \int_0^c \frac{AB}{c^2} y^2 \, dy$ $= \frac{1}{2} \left[\frac{AB}{3c^2} y^3 \right]_0^c$ $= \frac{1}{2} \left[\frac{ABC}{3} - 0 \right] = \frac{ABC}{6}$   	1 correct area 1 set up 1 progress 1 solution.	